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New Conditions for Boundedness of the Solution of a Matrix Riccati Differential Equation

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Sufficient conditions are given for a matrix Riccati differential equation to have a bounded solution; these conditions are less stringent than those known heretofore. The conditions are in the form of linear, algebraic and differential inequalities.

1. INTRODUCTION

A certain matrix Riccati differential equation appears often in optimal control theory; for example, [1], [2], [3], [4], [5]. In fact, the boundedness of the solution of this matrix Riccati equation is equivalent to the classical "no-conjugate-point condition" of the Calculus of Variations [2]. Apart from importance in the above mentioned area, the Riccati equation is intrinsically interesting [6]–[9] primarily because its solution can become unbounded in a finite time interval.

In this paper we present sufficient conditions for a matrix Riccati differential equation to have a bounded solution. The conditions are less restrictive than those known currently [1], [2].

Consider the following matrix Riccati differential equation:

$$-\frac{dS}{dt} = -\dot{S} = \tilde{Q} + \tilde{A}^T S + S\tilde{A} - (F + B^T S)^T R^{-1}(F + B^T S) \tag{1}$$

with boundary condition

$$S(t_f) = S_f \tag{2}$$

Assumptions. (i) The elements of the matrices $\tilde{Q}$, $\tilde{A}$, F, B and R^{-1} are continuous real functions of the continuous real variable t (time) in the interval $[t_0, t_f]$.

(ii) $\tilde{Q}(t)$ is nxn and symmetric,

(iii) $\tilde{A}(t)$ is nxn,

(iv) $F(t)$ is mxn,

(v) $B^T(t)$ is mxn,

(vi) R^{-1} is mxm, symmetric, and positive-definite $\forall t \in [t_0, t_f]$.

Conditions on these matrices are required which guarantee that $S(t)$; $t \in [t_0, t_f]$ is bounded.†

2. Known Sufficient Conditions for a Bounded Solution

It is known [1], [2] that the following conditions are sufficient to ensure that $S(t)$; $t \in [t_0, t_f]$ is bounded.

$$\text{(i)} \qquad \tilde{Q} - F^T R^{-1} F \geqslant 0 \qquad \forall t \in [t_0, t_f] \tag{3}$$

$$\text{(ii)} \qquad R^{-1} > 0 \qquad \forall t \in [t_0, t_f] \tag{4}$$

$$\text{(iii)} \qquad S_f \geqslant 0 \tag{5}$$

3. An Equivalent Riccati Differential Equation

Without loss of generality we shall consider the Riccati equation:

$$-\frac{dS}{dt} = Q + A^T S + SA - SBR^{-1}B^T S \tag{6}$$

$$S(t_f) = S_f \tag{7}$$

Equation (1) can be written in this form if we define

$$Q = \tilde{Q} - F^T R^{-1} F \tag{8}$$

$$A = \tilde{A} - BR^{-1}F \tag{9}$$

4. New Sufficient Conditions for a Bounded Solution

Theorem 1. *If there exists an nxn symmetric matrix function of time $P(t)$ whose elements are continuously differentiable real functions of t in the interval $[t_0, t_f]$ such that:*‡

$$\text{(i)} \qquad B^T P = 0 \qquad \forall t \in [t_0, t_f] \tag{10}$$

$$\text{(ii)} \qquad \dot{P} + Q + A^T P + PA = M(t) \geqslant 0 \qquad \forall t \in [t_0, t_f] \tag{11}$$

$$\text{(iii)} \qquad -P(t_f) + S_f = G_f \geqslant 0 \tag{12}$$

† Note that (1) is solved backward from the terminal time t_f.

‡ These conditions arise in the study of singular and nonsingular optimal control problems [10], [11].

where:

$M(t)$ is an nxn symmetric matrix function of time whose elements are continuous real functions in the interval $[t_0, t_f]$ and which is positive semi-definite, and where:

G_f is an nxn symmetric positive semi-definite matrix.
Then:

$$S(t); \quad t \in [t_0, t_f] \textit{ is bounded.} \tag{13}$$

Proof. Let

$$\bar{P}(t) + \bar{S}(t) = P(t); \qquad t \in [t_0, t_f] \tag{14}$$

where $\bar{P}$ and $\bar{S}$ are real symmetric matrix functions of time. Then, from (10), (11), and (14),

$$-\dot{\bar{P}} - \dot{\bar{S}} = Q + A^T(\bar{P} + \bar{S}) + (\bar{P} + \bar{S})A - M - (\bar{P} + \bar{S})BR^{-1}B^T(\bar{P} + \bar{S}) \tag{15}$$

$$\begin{aligned} &= Q + A^T(\bar{P} + \bar{S}) + (\bar{P} + \bar{S})A - M - \bar{S}BR^{-1}B^T\bar{S} - \bar{S}BR^{-1}B^T\bar{P} \\ &\quad -\bar{P}BR^{-1}B^T\bar{S} - \bar{P}BR^{-1}B^T\bar{P} \end{aligned} \tag{16}$$

Using (10) and (14) in (16), we obtain:

$$\begin{aligned} -\dot{\bar{P}} - \dot{\bar{S}} = Q &+ A^T(\bar{P} + \bar{S}) + (\bar{P} + \bar{S})A - M - \bar{S}BR^{-1}B^T\bar{S} \\ &+ \bar{P}BR^{-1}B^T\bar{P} \end{aligned} \tag{17}$$

Now, choose:

$$-\dot{\bar{P}} = -M + A^T\bar{P} + \bar{P}A + \bar{P}BR^{-1}B^T\bar{P} \tag{18}$$

From (12) and (14), we have that

$$-\bar{P}(t_f) - \bar{S}(t_f) + S_f = G_f \geqslant 0. \tag{19}$$

Choose:

$$\bar{P}(t_f) = -G_f\,. \tag{20}$$

From (18), (20), $\bar{P}(t)$; $t \in [t_0, t_f]$ is bounded because $(-\bar{P})$ satisfies a Riccati differential equation for which conditions (3)–(5) hold, viz.,

$$M(t) \geqslant 0 \qquad \forall t \in [t_0, t_f] \tag{21}$$

$$R^{-1}(t) > 0 \qquad \forall t \in [t_0, t_f] \tag{22}$$

$$G_f \geqslant 0 \tag{23}$$

Using (18) and (20) in (17) and (19) we obtain, finally:

$$-\dot{\bar{S}} = Q + A^T\bar{S} + \bar{S}A - \bar{S}BR^{-1}B^T\bar{S} \tag{24}$$

and

$$\bar{S}(t_f) = S_f \tag{25}$$

which is the same matrix Riccati equation as (6), (7).

Now since (10)–(12) are satisfied by a matrix function $P(t)$; $t \in [t_0, t_f]$ which has bounded elements, and since, by (18)–(23), $\bar{P}(t)$; $t \in [t_0, t_f]$ is bounded, it follows from (14) that $\bar{S}(t)$; $t \in [t_0, t_f]$ is bounded. This proves the Theorem.

THEOREM 2. *Conditions* (10)–(12) *of Theorem* 1 *are not more stringent than Conditions* (3)–(5).

Proof. Use (8), (9) in (10)–(12):

(i) $$B^TP = 0 \qquad \forall t \in [t_0, t_f] \tag{26}$$

(ii) $$\dot{P} + \tilde{Q} - F^TR^{-1}F + (\tilde{A} - BR^{-1}F)^T P + P(\tilde{A} - BR^{-1}F) = M(t) \geqslant 0 \qquad \forall t \in [t_0, t_f] \tag{27}$$

(iii) $$-P(t_f) + S_f = G_f \geqslant 0 \tag{28}$$

Clearly, if (3)–(5) are satisfied, then (26)–(28) are satisfied by:

$$P(t) = 0 = \dot{P}(t) \qquad \forall t \in [t_0, t_f] \tag{29}$$

This proves the Theorem.

5. EXAMPLE

In this Section we show by means of an example that conditions (10)–(12) are less restrictive than conditions (3)–(5).

$$n = 2, \quad M = 1, \quad t_0 = 0, \quad t_f = 1 \tag{30}$$

$$F = 0 \tag{31}$$

$$B^T = [0 \;\; 1] \tag{32}$$

$$R^{-1} = 1 \tag{33}$$

$$\tilde{Q} = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \tag{34}$$

$$\tilde{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{35}$$

$$S_f = 0 \tag{36}$$

Clearly, these values do *not* satisfy (3)–(5); i.e., the known sufficiency conditions are violated.

Conditions (10)–(12) become:

$$[P_{12} \quad P_{22}] = 0 \qquad \forall t \in [0, 1] \tag{37}$$

$$\begin{bmatrix} \dot{P}_{11} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & P_{11} \\ P_{11} & 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \geqslant 0; \qquad \forall t \in [0, 1] \tag{38}$$

and

$$\begin{bmatrix} -P_{11}(1) & 0 \\ 0 & 0 \end{bmatrix} \geqslant 0 \tag{39}$$

Let us choose $P_{11}(1) = 0$; this satisfies (39). If we choose

$$\dot{P}_{11} = 2 \qquad \text{then} \qquad P_{11}(0) = -2,$$

and (38) becomes:

$$\begin{bmatrix} 1 & -2 + 2t \\ -2 + 2t & 4 \end{bmatrix} \geqslant 0 \tag{41}$$

Ineq. (41) holds $\forall t \in [0, 1]$. Thus the Riccati equation associated with the values of F, B, R^{-1}, $\tilde{Q}$, $\tilde{A}$ and S_f given by (30)–(36) has a bounded solution in the interval [0, 1], even though conditions (3)–(5) are not satisfied.

6. Conclusion

Boundedness of solution of a certain matrix Riccati differential equation is shown to be guaranteed if the linear algebraic and differential inequalities (10)–(12) hold. An example is used to illustrate that these conditions are less restrictive than those known heretofore [1], [2]. Conditions (10)–(12) arise in the study of singular and nonsingular optimal control problems [10], [11].

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